

A NonLocal Epidemic Model
with
Age Structure, Space Dependence and Quarantine

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ECMI, 29.04.2020

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A NonLocal Epidemic Model

The Model

Qualitative Properties

- Age Dependence

- Density Dependence

- Care Homes

- A Football Match

- The Relevance of Quarentine

Open Problems

A NonLocal Epidemic Model

(Kermack, McKendrick: Proc. Royal Soc. A, 1927)

Susceptible

$$S = S(t)$$

Infected

$$I = I(t)$$

Recovered

$$R = R(t)$$

$$\begin{cases} \dot{S} = \text{mortality} + \text{interaction} \\ \dot{I} = \text{mortality} + \text{interaction} \\ \dot{R} = \text{mortality} + \text{interaction} \end{cases}$$

A NonLocal Epidemic Model

(Kermack, McKendrick: Proc. Royal Soc. A, 1927)

With **Quarantine**

Susceptible

$$S = S(t)$$

Infected

$$I = I(t)$$

Recovered

$$R = R(t)$$

Susceptible

$$S = S(t)$$

Infective

$$I = I(t)$$



Hospitalized

$$H = H(t)$$

Recovered

$$R = R(t)$$

$$\left\{ \begin{array}{l} \dot{S} = \text{mortality} + \text{interaction} \\ \dot{i} = \text{mortality} + \text{interaction} \\ \dot{H} = \text{mortality} + \text{interaction} \\ \dot{R} = \text{mortality} + \text{interaction} \end{array} \right.$$

A NonLocal Epidemic Model – $a \in \mathbb{R}_+$

(Kermack, McKendrick: Proc. Royal Soc. A, 1927)

With Quarantine, **Age Structure**

Susceptible

$$S = S(t)$$

Infected

$$I = I(t)$$

Recovered

$$R = R(t)$$

Susceptible

$$S = S(t, a)$$

Infective

$$I = I(t, a)$$



Hospitalized

$$H = H(t, a)$$

Recovered

$$R = R(t, a)$$

$$\left\{ \begin{array}{l} \partial_t S + \partial_a S = \text{mortality} + \text{interaction} \\ \partial_t I + \partial_a I = \text{mortality} + \text{interaction} \\ \partial_t H + \partial_a H = \text{mortality} + \text{interaction} \\ \partial_t R + \partial_a R = \text{mortality} + \text{interaction} \end{array} \right.$$

A NonLocal Epidemic Model – $a \in \mathbb{R}_+$, $x \in X$

(Kermack, McKendrick: Proc. Royal Soc. A, 1927)

With Quarantine, Age Structure and **Space Dependence**

Susceptible

$$S = S(t)$$

Infected

$$I = I(t)$$

Recovered

$$R = R(t)$$



Susceptible

$$S = S(t, a, x)$$

Infective

$$I = I(t, a, x)$$

Hospitalized

$$H = H(t, a, x)$$

Recovered

$$R = R(t, a, x)$$

$$v_S = v_S(t, a, x) \quad v_I = v_I(t, a, x)$$

$$v_R = v_R(t, a, x)$$

$$\left\{ \begin{array}{l} \partial_t S + \partial_a S + \operatorname{div}_x (v_S S) = \text{mortality} + \text{interaction} \\ \partial_t I + \partial_a I + \operatorname{div}_x (v_I I) = \text{mortality} + \text{interaction} \\ \partial_t H + \partial_a H = \text{mortality} + \text{interaction} \\ \partial_t R + \partial_a R + \operatorname{div}_x (v_R R) = \text{mortality} + \text{interaction} \end{array} \right.$$

A NonLocal Epidemic Model – $a \in \mathbb{R}_+$, $x \in X$

(Colombo, Garavello, Marcellini, Rossi: In preparation, 2020)

With Quarantine, Age Structure and Space Dependence

Susceptible

$$S = S(t)$$

Infected

$$I = I(t)$$

Recovered

$$R = R(t)$$

Susceptible

$$S = S(t, a, x)$$

Infective

$$I = I(t, a, x)$$

Hospitalized

$$H = H(t, a, x)$$

Recovered

$$R = R(t, a, x)$$

$$v_S = v_S(t, a, x) \quad v_I = v_I(t, a, x)$$

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$$\left\{ \begin{array}{l} \partial_t S + \partial_a S + \operatorname{div}_x (v_S S) = \text{mortality} + \text{interaction} \\ \partial_t I + \partial_a I + \operatorname{div}_x (v_I I) = \text{mortality} + \text{interaction} \\ \partial_t H + \partial_a H = \text{mortality} + \text{interaction} \\ \partial_t R + \partial_a R + \operatorname{div}_x (v_R R) = \text{mortality} + \text{interaction} \end{array} \right.$$

A NonLocal Epidemic Model – Mortality

$$\left\{ \begin{array}{l} \partial_t S + \partial_a S + \operatorname{div}_x (v_S S) = \text{mortality} + \text{interaction} \\ \partial_t I + \partial_a I + \operatorname{div}_x (v_I I) = \text{mortality} + \text{interaction} \\ \partial_t H + \partial_a H = \text{mortality} + \text{interaction} \\ \partial_t R + \partial_a R + \operatorname{div}_x (v_R R) = \text{mortality} + \text{interaction} \end{array} \right.$$

A NonLocal Epidemic Model – Mortality

Proportional to the population,
with a coefficient time, age and space dependent

$$\left\{ \begin{array}{l} \partial_t S + \partial_a S + \operatorname{div}_x (v_S S) + \mu_S S = \text{interaction} \\ \partial_t I + \partial_a I + \operatorname{div}_x (v_I I) + \mu_I I = \text{interaction} \\ \partial_t H + \partial_a H + \mu_H H = \text{interaction} \\ \partial_t R + \partial_a R + \operatorname{div}_x (v_R R) + \mu_R R = \text{interaction} \end{array} \right.$$

$$\begin{aligned} \mu_S &= \mu_S(t, a, x), & \mu_I &= \mu_I(t, a, x) \\ \mu_H &= \mu_H(t, a, x), & \mu_R &= \mu_R(t, a, x) \end{aligned}$$

A NonLocal Epidemic Model – Interactions

$$\left\{ \begin{array}{l} \partial_t S + \partial_a S + \operatorname{div}_x (v_S S) + \mu_S S = \text{interaction} \\ \partial_t I + \partial_a I + \operatorname{div}_x (v_I I) + \mu_I I = \text{interaction} \\ \partial_t H + \partial_a H + \mu_H H = \text{interaction} \\ \partial_t R + \partial_a R + \operatorname{div}_x (v_R R) + \mu_R R = \text{interaction} \end{array} \right.$$

A NonLocal Epidemic Model – Interactions

$$\left\{ \begin{array}{l} \partial_t S + \partial_a S + \operatorname{div}_x (v_S S) + \mu_S S = -(S \rightarrow I) \\ \partial_t I + \partial_a I + \operatorname{div}_x (v_I I) + \mu_I I = -(I \rightarrow H) - (I \rightarrow R) + (S \rightarrow I) \\ \partial_t H + \partial_a H + \mu_H H = (I \rightarrow H) - (H \rightarrow R) \\ \partial_t R + \partial_a R + \operatorname{div}_x (v_R R) + \mu_R R = (I \rightarrow R) + (H \rightarrow R) \end{array} \right.$$

A NonLocal Epidemic Model – Interactions

$$\left\{ \begin{array}{l} \partial_t S + \partial_a S + \operatorname{div}_x (v_S S) + \mu_S S = -(S \rightarrow I) \\ \partial_t I + \partial_a I + \operatorname{div}_x (v_I I) + \mu_I I = -\kappa I - (I \rightarrow R) + (S \rightarrow I) \\ \partial_t H + \partial_a H + \mu_H H = \kappa I - (H \rightarrow R) \\ \partial_t R + \partial_a R + \operatorname{div}_x (v_R R) + \mu_R R = (I \rightarrow R) + (H \rightarrow R) \end{array} \right.$$

$$\kappa = \kappa(t, a, x)$$

$(I \rightarrow H) = \kappa I$ rate at which infective individuals are confined

A NonLocal Epidemic Model – Interactions

$$\begin{cases} \partial_t S + \partial_a S + \operatorname{div}_x (v_S S) + \mu_S S = -(S \rightarrow I) \\ \partial_t I + \partial_a I + \operatorname{div}_x (v_I I) + \mu_I I = -\kappa I - \vartheta I + (S \rightarrow I) \\ \partial_t H + \partial_a H + \mu_H H = \kappa I - (H \rightarrow R) \\ \partial_t R + \partial_a R + \operatorname{div}_x (v_R R) + \mu_R R = \vartheta I + (H \rightarrow R) \end{cases}$$

$$\kappa = \kappa(t, a, x), \quad \vartheta = \vartheta(t, a, x)$$

$(I \rightarrow H) = \kappa I$ rate at which infective individuals are confined

$(I \rightarrow R) = \vartheta I$ rate at which infective individuals recover

A NonLocal Epidemic Model – Interactions

$$\begin{cases} \partial_t S + \partial_a S + \operatorname{div}_x (v_S S) + \mu_S S = -(S \rightarrow I) \\ \partial_t I + \partial_a I + \operatorname{div}_x (v_I I) + \mu_I I = -\kappa I - \vartheta I + (S \rightarrow I) \\ \partial_t H + \partial_a H + \mu_H H = \kappa I - \eta H \\ \partial_t R + \partial_a R + \operatorname{div}_x (v_R R) + \mu_R R = \vartheta I + \eta H \end{cases}$$

$$\kappa = \kappa(t, a, x), \quad \vartheta = \vartheta(t, a, x), \quad \eta = \eta(t, a, x)$$

$(I \rightarrow H) = \kappa I$ rate at which infective individuals are confined

$(I \rightarrow R) = \vartheta I$ rate at which infective individuals recover

$(H \rightarrow R) = \eta H$ rate at which hospitalized individuals recover

A NonLocal Epidemic Model – Interactions

$$\begin{cases} \partial_t S + \partial_a S + \operatorname{div}_x (v_S S) + \mu_S S = -(S \rightarrow I) \\ \partial_t I + \partial_a I + \operatorname{div}_x (v_I I) + \mu_I I = -\kappa I - \vartheta I + (S \rightarrow I) \\ \partial_t H + \partial_a H + \mu_H H = \kappa I - \vartheta H \\ \partial_t R + \partial_a R + \operatorname{div}_x (v_R R) + \mu_R R = \vartheta I + \vartheta H \end{cases}$$

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$(I \rightarrow H) = \kappa I$ rate at which infective individuals are confined

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$(H \rightarrow R) = \eta H$ rate at which hospitalized individuals recover

$(S \rightarrow I)$ rate at which susceptible individuals are infected

A NonLocal Epidemic Model – Interactions

$$\begin{cases} \partial_t S + \partial_a S + \operatorname{div}_x (v_S S) + \mu_S S = -(S \rightarrow I) \\ \partial_t I + \partial_a I + \operatorname{div}_x (v_I I) + \mu_I I = -\kappa I - \vartheta I + (S \rightarrow I) \\ \partial_t H + \partial_a H + \mu_H H = \kappa I - \vartheta H \\ \partial_t R + \partial_a R + \operatorname{div}_x (v_R R) + \mu_R R = \vartheta I + \vartheta H \end{cases}$$

$$\kappa = \kappa(t, a, x), \quad \vartheta = \vartheta(t, a, x), \quad \eta = \eta(t, a, x)$$

$(I \rightarrow H) = \kappa I$ rate at which infective individuals are confined

$(I \rightarrow R) = \vartheta I$ rate at which infective individuals recover

$(H \rightarrow R) = \eta H$ rate at which hospitalized individuals recover

$(S \rightarrow I)$ rate at which susceptible individuals are infected

$$(S \rightarrow I) = \int_{\mathbb{R}_+} \int_X \rho(t, a, \alpha, x, \xi) I(t, \alpha, \xi) d\xi d\alpha S(t, a, x)$$

A NonLocal Epidemic Model – Interactions

$$\begin{cases} \partial_t S + \partial_a S + \operatorname{div}_x (v_S S) + \mu_S S = -(S \rightarrow I) \\ \partial_t I + \partial_a I + \operatorname{div}_x (v_I I) + \mu_I I = -\kappa I - \vartheta I + (S \rightarrow I) \\ \partial_t H + \partial_a H + \mu_H H = \kappa I - \vartheta H \\ \partial_t R + \partial_a R + \operatorname{div}_x (v_R R) + \mu_R R = \vartheta I + \vartheta H \end{cases}$$

$$\kappa = \kappa(t, a, x), \quad \vartheta = \vartheta(t, a, x), \quad \eta = \eta(t, a, x)$$

$(I \rightarrow H) = \kappa I$ rate at which infective individuals are confined

$(I \rightarrow R) = \vartheta I$ rate at which infective individuals recover

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$$(S \rightarrow I) = \int_{\mathbb{R}_+} \int_X \rho(t, a, \alpha, x, \xi) I(t, \alpha, \xi) d\xi d\alpha S(t, a, x)$$

(Colombo, Garavello, Marcellini, Rossi: In preparation, 2020)

A NonLocal Epidemic Model – Interactions

$$\begin{cases} \partial_t S + \partial_a S + \operatorname{div}_x (v_S S) + \mu_S S = -(S \rightarrow I) \\ \partial_t I + \partial_a I + \operatorname{div}_x (v_I I) + \mu_I I = -\kappa I - \vartheta I + (S \rightarrow I) \\ \partial_t H + \partial_a H + \mu_H H = \kappa I - \vartheta H \\ \partial_t R + \partial_a R + \operatorname{div}_x (v_R R) + \mu_R R = \vartheta I + \vartheta H \end{cases}$$

$$\kappa = \kappa(t, a, x), \quad \vartheta = \vartheta(t, a, x), \quad \eta = \eta(t, a, x)$$

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$(S \rightarrow I)$ rate at which susceptible individuals are infected

$$(S \rightarrow I) = \int_{\mathbb{R}_+} \int_{\mathcal{X}} \rho(t, a, \alpha, x, \xi) I(t, \alpha, \xi) d\xi d\alpha S(t, a, x)$$

(Hopefully, no $(R \rightarrow S)$ term!)

A NonLocal Epidemic Model – Role of ρ

$$\begin{cases} \partial_t S + \partial_a S + \operatorname{div}_x (v_S S) + \mu_S S & = - \iint \rho I \, d\alpha \, d\xi S \\ \partial_t I + \partial_a I + \operatorname{div}_x (v_I I) + \mu_I I & = \iint \rho I \, d\alpha \, d\xi S - \kappa I - \vartheta I \\ \partial_t H + \partial_a H + \mu_H H & = \kappa I - \eta H \\ \partial_t R + \partial_a R + \operatorname{div}_x (v_R R) + \mu_R R & = \vartheta I + \eta H \end{cases}$$

$$(S \rightarrow I) = \int_{\mathbb{R}_+} \int_X \rho(t, a, \alpha, x, \xi) I(t, \alpha, \xi) \, d\xi \, d\alpha \, S(t, a, x)$$

ρ describes the disease transmission

$$\text{e.g.: } \|x - \xi\| > \delta \Rightarrow \rho = 0$$

A NonLocal Epidemic Model – Role of ρ and κ

$$\left\{ \begin{array}{l} \partial_t S + \partial_a S + \operatorname{div}_x (v_S S) + \mu_S S = - \iint \rho I \, d\alpha \, d\xi S \\ \partial_t I + \partial_a I + \operatorname{div}_x (v_I I) + \mu_I I = \iint \rho I \, d\alpha \, d\xi S - \kappa I - \vartheta I \\ \partial_t H + \partial_a H + \mu_H H = \kappa I - \eta H \\ \partial_t R + \partial_a R + \operatorname{div}_x (v_R R) + \mu_R R = \vartheta I + \eta H \end{array} \right.$$

$$(S \rightarrow I) = \int_{\mathbb{R}_+} \int_X \rho(t, a, \alpha, x, \xi) I(t, \alpha, \xi) \, d\xi \, d\alpha \, S(t, a, x)$$

ρ describes the disease transmission

κ describes the isolation of infective individuals

A NonLocal Epidemic Model – Computing R_0

Nondimensional, time dependent, independent of a and x :

$$R_0(t) = \frac{\text{(average infection rate} \times \text{number of susceptibles)}}{\text{(average recovery rate)}}$$

A NonLocal Epidemic Model – Computing R_o

Nondimensional, time dependent, independent of a and x :

$$R_o(t) = \frac{\text{(average infection rate} \times \text{number of susceptibles)}}{\text{(average recovery rate)}}$$

$$R_o(t) = \frac{\frac{\int \rho I S}{\int I}}{\frac{\int \vartheta I + \int \eta H}{\int I + \int H}}$$

A NonLocal Epidemic Model – Computing R_o

Nondimensional, time dependent, independent of a and x :

$$R_o(t) = \frac{\text{(average infection rate} \times \text{number of susceptibles)}}{\text{(average recovery rate)}}$$

$$R_o(t) = \frac{\frac{\iiint \rho(t, a, \alpha, x, \xi) I(t, \alpha, \xi) S(t, a, x) d\xi d\alpha dx da}{\iint I(t, a, x) dx da}}{\frac{\iint \vartheta(t, a, x) I(t, a, x) dx da + \iint \eta(t, a, x) H(t, a, x) dx da}{\iint I(t, a, x) da dx + \iint H(t, a, x) da dx}}$$

A NonLocal Epidemic Model – Numerical Integrations

$$\begin{cases} \partial_t S + \partial_a S + \operatorname{div}_x(v_S S) + \mu_S S & = - \iint \rho I d\alpha d\xi S \\ \partial_t I + \partial_a I + \operatorname{div}_x(v_I I) + \mu_I I & = \iint \rho I d\alpha d\xi S - \kappa I - \vartheta I \\ \partial_t H + \partial_a H + \mu_H H & = \kappa I - \eta H \\ \partial_t R + \partial_a R + \operatorname{div}_x(v_R R) + \mu_R R & = \vartheta I + \eta H \end{cases}$$

Neglect the aging terms $\partial_a S, \partial_a I, \partial_a H, \partial_a R$ on short time intervals

Neglect the movement terms $\operatorname{div}_x(v_S S), \operatorname{div}_x(v_I I), \operatorname{div}_x(v_R R)$

A NonLocal Epidemic Model – Numerical Integrations

$$\begin{cases} \partial_t S + \partial_a S + \operatorname{div}_x(v_S S) + \mu_S S & = - \iint \rho I d\alpha d\xi S \\ \partial_t I + \partial_a I + \operatorname{div}_x(v_I I) + \mu_I I & = \iint \rho I d\alpha d\xi S - \kappa I - \vartheta I \\ \partial_t H + \partial_a H + \mu_H H & = \kappa I - \eta H \\ \partial_t R + \partial_a R + \operatorname{div}_x(v_R R) + \mu_R R & = \vartheta I + \eta H \end{cases}$$

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Neglect the movement terms $\operatorname{div}_x(v_S S), \operatorname{div}_x(v_I I), \operatorname{div}_x(v_R R)$

$$\begin{cases} \dot{S} + \mu_S S & = - \iint \rho I d\alpha d\xi S \\ \dot{I} + \mu_I I & = \iint \rho I d\alpha d\xi S - \kappa I - \vartheta I \\ \dot{H} + \mu_H H & = \kappa I - \eta H \\ \dot{R} + \mu_R R & = \vartheta I + \eta H \end{cases}$$

A NonLocal Epidemic Model – Numerical Integrations

$$\begin{cases} \partial_t S + \partial_a S + \operatorname{div}_x(v_S S) + \mu_S S &= - \iint \rho I \, d\alpha \, d\xi S \\ \partial_t I + \partial_a I + \operatorname{div}_x(v_I I) + \mu_I I &= \iint \rho I \, d\alpha \, d\xi S - \kappa I - \vartheta I \\ \partial_t H + \partial_a H + \mu_H H &= \kappa I - \eta H \\ \partial_t R + \partial_a R + \operatorname{div}_x(v_R R) + \mu_R R &= \vartheta I + \eta H \end{cases}$$

Neglect the aging terms $\partial_a S, \partial_a I, \partial_a H, \partial_a R$ on short time intervals

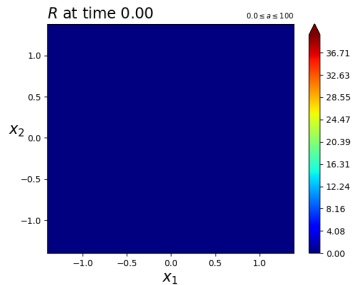
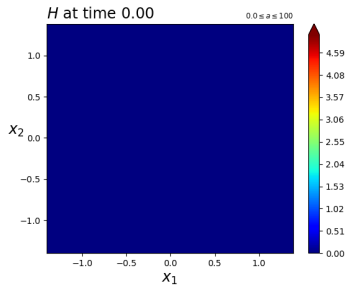
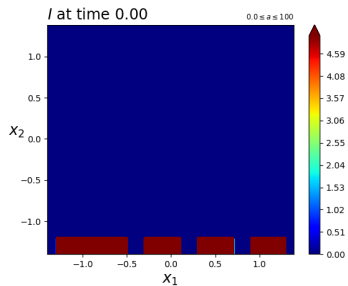
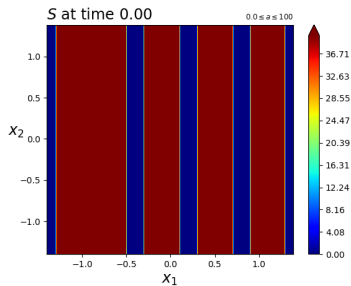
Neglect the movement terms $\operatorname{div}_x(v_S S), \operatorname{div}_x(v_I I), \operatorname{div}_x(v_R R)$

$$\begin{cases} \dot{S} + \mu_S S &= - \iint \rho I \, d\alpha \, d\xi S \\ \dot{I} + \mu_I I &= \iint \rho I \, d\alpha \, d\xi S - \kappa I - \vartheta I \\ \dot{H} + \mu_H H &= \kappa I - \eta H \\ \dot{R} + \mu_R R &= \vartheta I + \eta H \end{cases}$$

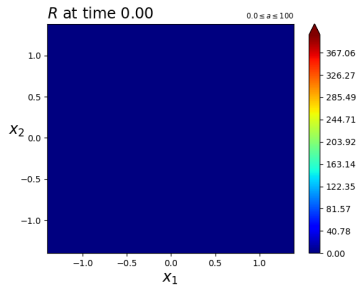
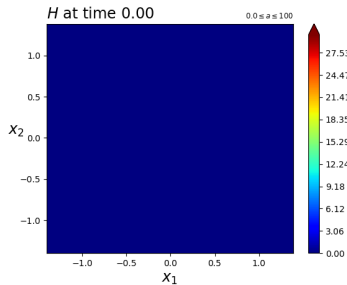
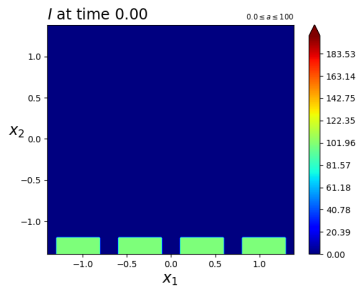
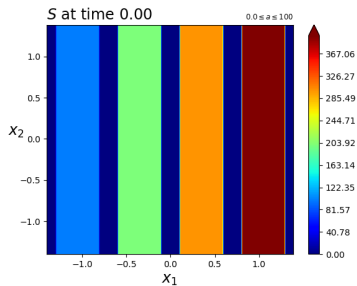
with parameters “*inspired*” by Italian data

4 Age classes: $a < 40$, $40 \leq a < 60$, $60 \leq a < 80$, $80 < a$.

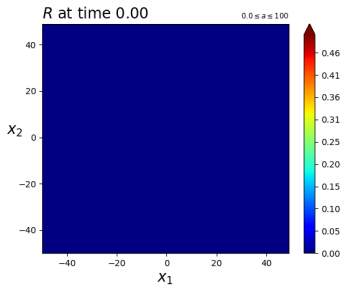
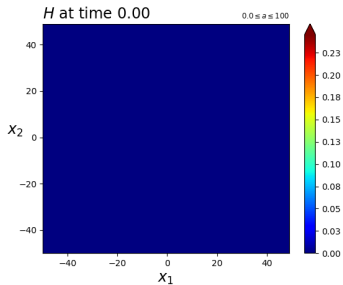
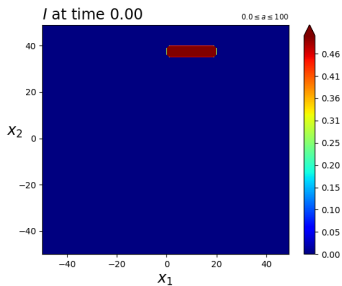
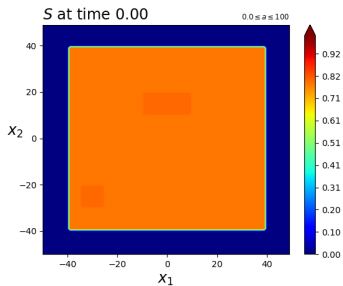
A NonLocal Epidemic Model – Age



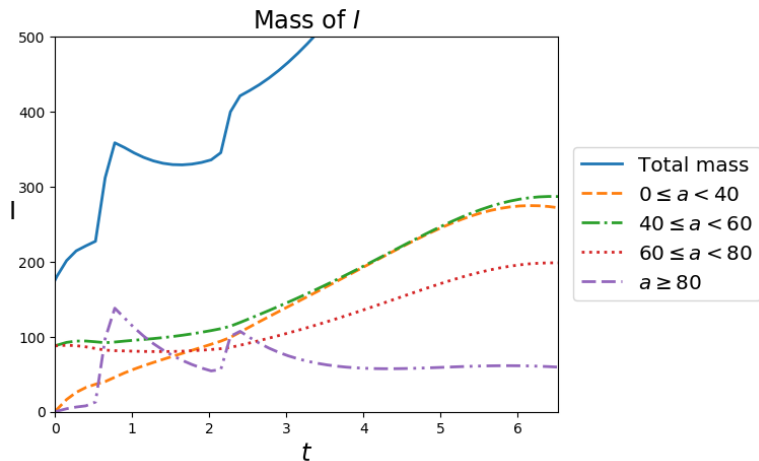
A NonLocal Epidemic Model – Density



A NonLocal Epidemic Model – Care Homes

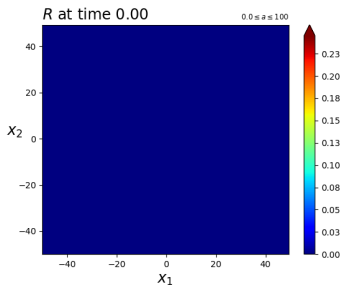
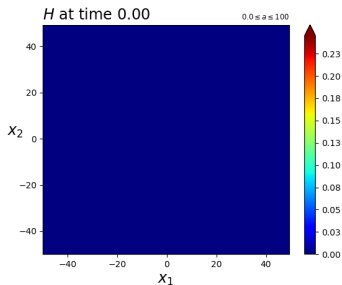
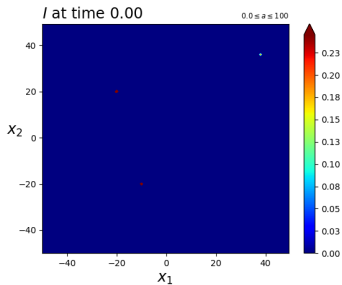
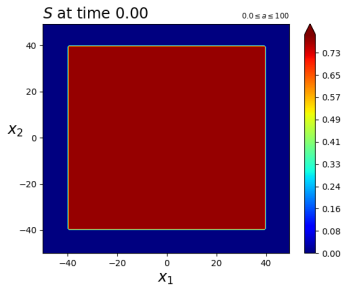


A NonLocal Epidemic Model – Care Homes



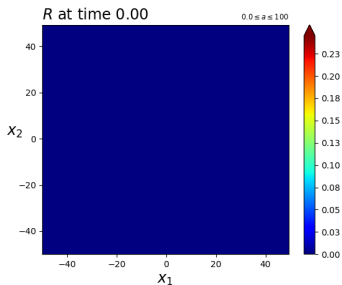
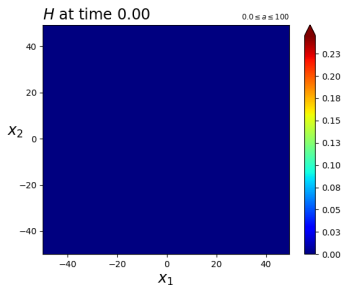
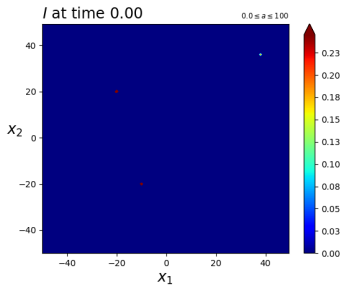
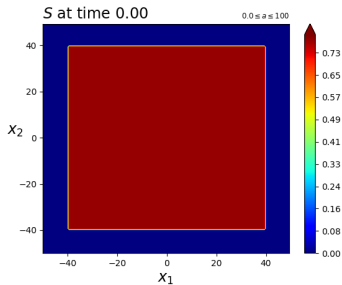
A NonLocal Epidemic Model – A Football Match

No Match



A NonLocal Epidemic Model – A Football Match

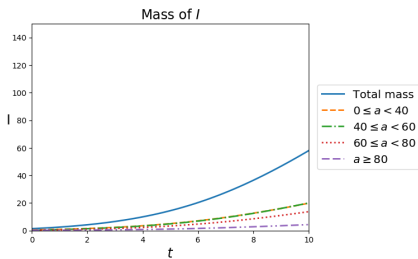
Match for $t \in [0.2, 0.7]$



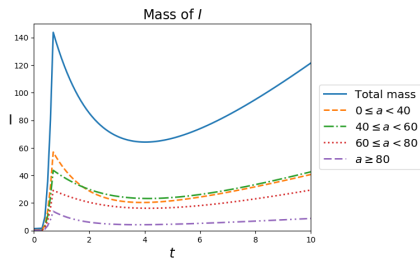
A NonLocal Epidemic Model – A Football Match

I totals

No Match



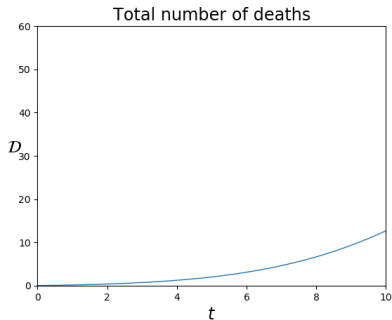
With Match



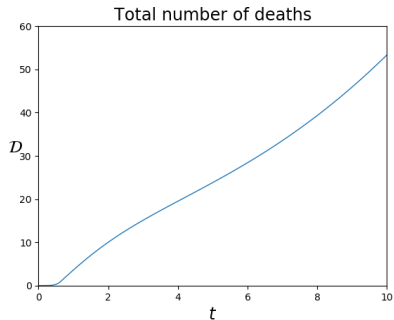
A NonLocal Epidemic Model – A Football Match

Deaths due to the virus

No Match

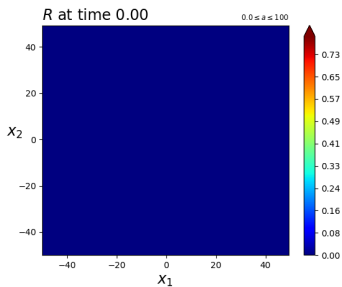
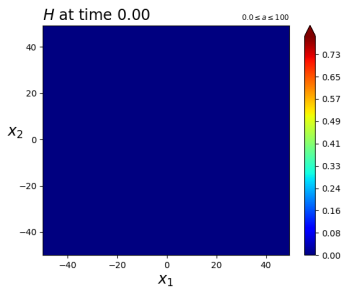
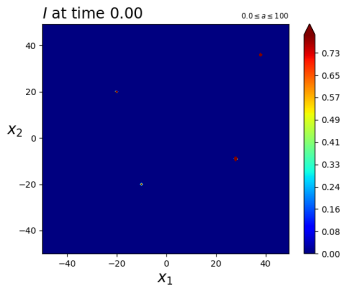
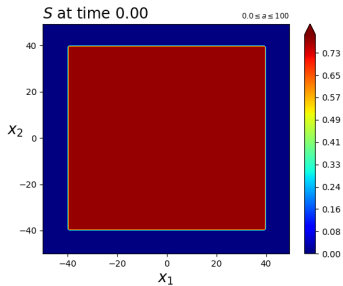


With Match



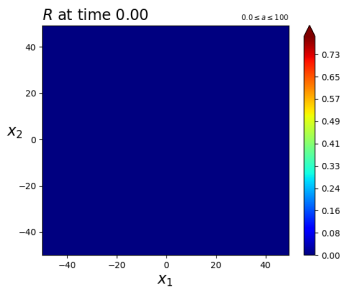
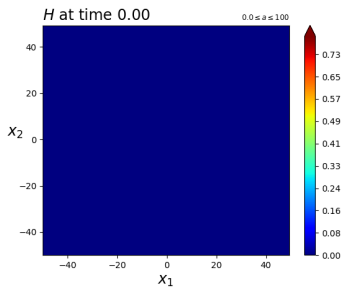
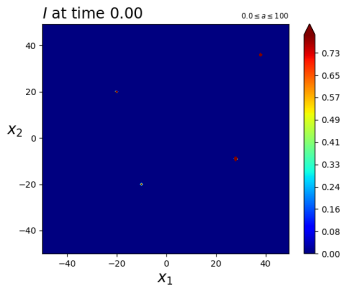
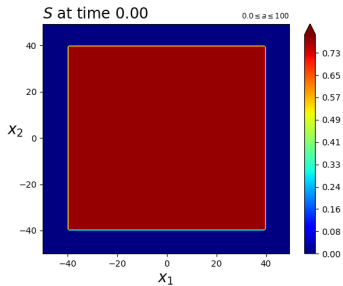
The Relevance of Quarentine

“Low” κ



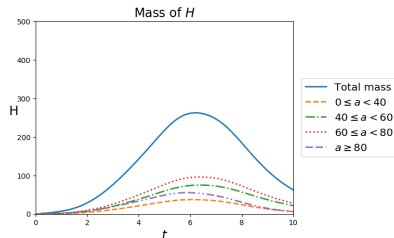
The Relevance of Quarentine

“High” κ

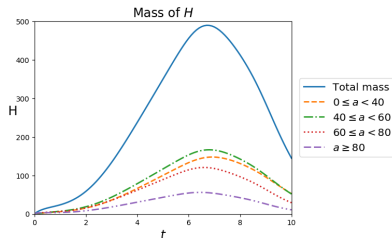


The Relevance of Quarantine

“Low” κ

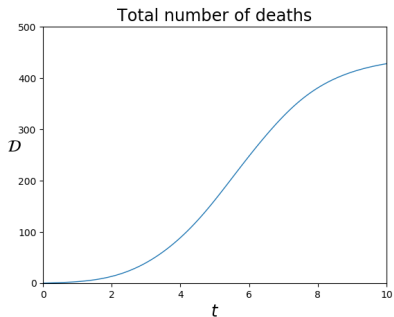


“High” κ

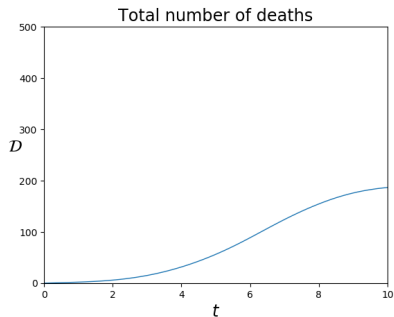


The Relevance of Quarantine

“Low” κ



“High” κ



Open Problems

- ▶ Well Posedness, Stability and Qualitative Properties
- ▶ Propagation Speed
- ▶ *Ad hoc* numerical algorithms
- ▶ Comparing Quarantine Strategies
Colombo, Garavello
Well Posedness and Control in a NonLocal SIR Model
AMOP, To appear
- ▶ Optimal Vaccination Policies
Colombo, Garavello
Optimizing Vaccination Strategies in an Age Structured SIR Model
MBE, 2020